

```
1 import numpy as np
2
3 import sympy
4 import matplotlib.pyplot as plt
5 plt.rcParams.update({'font.size': 22})
6
7 import seaborn as sns
8 sns.set_style("darkgrid")
9
10 import sys, os
```

This code block generates an  $N = m \times n$  matrix of spin values (-1 or 1). Then it calculates the magnetization of that matrix. By using the magnetization of a large sample of matrices, we can find the dispersion.

The 2D Ising Model:  
 $E(r) = -J \sum_{ij} \sigma_i \sigma_j + \mu_B H \sum_i \sigma_i$   
 here each  $\sigma_i \sigma_j$  is a pair of nearest neighbors.  
 Magnetization:  
 $m = \frac{1}{N} \sum_{i=1}^m \sum_{j=1}^n \sigma_{i,j}$   
 Dispersion:  
 $\overline{\Delta m^2} = \overline{m^2} - \overline{m}^2$

```
1 def Generate_Spins(N):
2     matrix = np.random.randint(2,size=(N))
3     matrix[matrix == 0] = -1
4     magnetization = np.sum(matrix)/np.prod(N)
5     return {"matrix":matrix, "mag":magnetization}

three_ex = Generate_Spins(3,3)
print("Matrix: \n", three_ex["matrix"], "\n \n Magntization: \n", three_ex["mag"])
```

```
Matrix:
[[ 1  1 -1]
 [-1  1 -1]
 [ 1  1  1]]

Magnitization:
0.3333333333333333
```

```
1 ten_ex = Generate_Spins((10,10))
2 print("Matrix: \n", ten_ex["matrix"], "\n \n Magntization: \n", ten_ex["mag"])
```

```
Matrix:
[[-1 1 1 1 1 -1 -1 1 1 -1 1]
 [1 1 -1 -1 -1 -1 -1 1 1 -1 1]
 [1 -1 -1 1 -1 1 -1 1 1 1 1]
 [1 -1 1 1 -1 1 -1 -1 1 -1 1]
 [1 1 1 -1 1 1 1 1 1 -1 -1]
 [1 1 -1 -1 -1 -1 -1 1 -1 1 1]
 [1 -1 -1 -1 -1 1 1 1 -1 1 1]
 [-1 1 1 1 1 -1 -1 -1 -1 1 1]
 [-1 -1 -1 1 1 -1 -1 -1 1 1 1]]

Magnetization:
-0.06
```

```
1 big_ex = Generate_Spins((64,64))
2 print("Matrix: \n", big_ex["matrix"], "\n \n Magntization: \n", big_ex["mag"])
```

```
Matrix:
[[ 1 1 1 ... 1 -1 1]
 [ 1 -1 1 ... -1 -1 -1]
 [ 1 1 -1 ... -1 1 1]
 ...
 [-1 -1 1 ... -1 -1 -1]
 [-1 1 1 ... -1 1 1]
 [-1 1 -1 ... 1 1 -1]]

Magnetization:
-0.0146484375
```

We will use a for loop to generate 50000  $771 \times 71$  matrices and estimate the dispersion.

```

1 def Calc_Dispersion(N,steps):
2     m_squared_bar = m_bar = 0
3
4     for idx in range(0,steps):
5         m = Generate_Spins(N)[ "mag" ]
6         m_bar += m; m_squared_bar += m**2
7
8     for item in (m_bar,m_squared_bar): item = item/steps
9
10    return {"dispersion": m_squared_bar - m_bar**2,
11            "m_squared_bar": m_squared_bar,
12            "m_bar": m_bar,}

```

```
1 N, steps = (3,3), 100000
2 print(sorted(Calc_Dispersion(N,steps).items()))

[('dispersion', 9856.148148148499), ('m bar', 35.77777777777782), ('m_squared bar', 11136.197530864494)]
```

```
1 N, steps = (3,3), 100000
2 print(sorted(Calc_Dispersion(N,steps).items()))

[('dispersion', -21988.5432089876087), ('n_bar', 182.22222222222852), ('n_squared_bar', 11216.395061728232)]
```

```
1 N,steps = (10,10), 100000
2 print(sorted(Calc_Dispersion(N,steps).items()))
```

```
1 N,steps = (10,10), 100000
2 print(sorted(Calc_Dispersion(N,steps).items()))

[('dispersion', 1006.8064000006458), ('m_bar', 0.019999999999858665), ('m_squared_bar', 1006.80680000006458)]
```

```
1 N_steps = (64,64), 100000
2 print(sorted(Calc_Dispersion(N_steps).items()))

[('dispersion', 8.82074928236914), ('m_bar', 3.94180453125), ('m_squared_bar', 24.359281778335571)]
```

```
1 N,steps = (64,64), 100000
2 print(sorted(Calc_Dispersion(N,steps).items()))

[('dispersion', -142.68939953232364), ('n bar', 12.919921875), ('n squared bar', 24.320441722869873)]
```

Average Energy:  

$$\bar{E} = \frac{1}{N_{MC}} \sum_{r=1}^{N_{MC}} E(r)$$
Average Magnetization:  

$$\bar{M} = \frac{1}{N_{MC}} \sum_{r=1}^{N_{MC}} M(r)$$
Difference in Energy if Flipping a Single Spin:  

$$\Delta E_{2D} = 2J\sigma_{i,j}(\sigma_{i-1,j} + \sigma_{i+1,j} + \sigma_{i,j-1} + \sigma_{i,j+1})$$

```

1 def Energy_Difference(M,x,y,x_max,y_max):
2     dE = 2*M[x,y]*[M[(x-1)%x_max,y]+M[(x+1)%x_max,y]+
3         M[x,(y-1)%y_max]+M[x,(y+1)%y_max]]
4     return dE
5
6 def Compute_Energy(M,x_max,y_max):
7     E = 0
8     for x in range(0,x_max):
9         for y in range(0,y_max):
10             E -= M[x,y]*[M[(x-1)%x_max,y]+M[x,(y+1)%y_max]]
11     return E

```

```

1 def Metropolis_Sampling(N,steps,beta):
2     x_max,y_max = 0; idx = N; idx = E_idx = 0; bar_m = []; bar_E = []
3
4     #1 Generate Spins
5     spins = Generate_Spins(N)[“matrix”]
6     while idx < steps:
7         idx += 1
8
9         #2a Pick a spin at random
10        x,y = np.random.randint(0,x_max),np.random.randint(0,y_max)
11
12        #2b Calculate Energy difference
13        dE = Energy_difference(spins,x,y,x_max,y_max)
14
15        #3a If dE <= 0, flip spin
16        if dE <= 0:
17            spins[x][y] = -1
18
19        #3b Else, if a random x ~ e-beta dE flip spin
20        elif (np.log(np.random.uniform(0,1)) < -beta*dE) flip spin

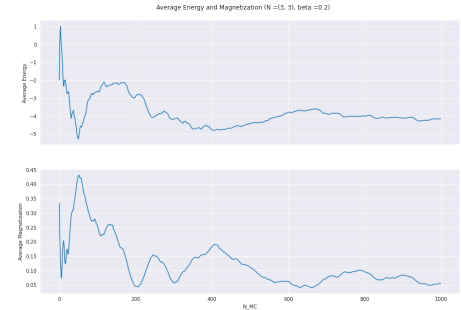
```

```
spins[i,y]
#3c if not, keep in unchanged
#3d Calculate barE
E = Compute_Energy(spins,x_max,y_max)
E_idx += E; bar_E.append(E_idx/idx)
#4b Calculate barM
M_idx += np.sum(spins)/np.prod(N); bar_M.append(M_idx/idx)
#5 Providing an update throughout the cycle
if idx % (steps*0.1) == 0:
    print("*****\n Iterations:",
          idx, " ("+"{:0%}".format(idx/steps), "complete)",
          "\n Average Energy:", bar_E[idx-1],
          "\n Average Magnetization:", bar_M[idx-1],
          "\n*****")
return ("bar_E":bar_E, "bar_M":bar_M, "beta":beta, "dim":N)
```

```
def Make_Plots(data):
    fig, axs = plt.subplots(2, sharex=True, figsize=(15,10))
    fig.suptitle("Average Energy and Magnetization (N = "+ str(data["dim"]) +
    " , beta = "+ str(data["beta"])+
    "+")", y=0.925)
    axs[0].plot(data["bar_E"]); axs[0].set(ylabel="Average Energy")
    axs[1].plot(data["bar_M"]); axs[1].set(ylabel="Average Magnetization")
    axs[1].set(xlabel="N_MC")
```

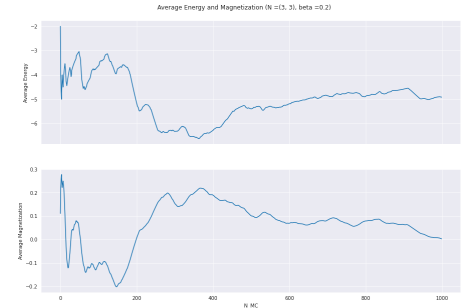
```
1 N = (3,3); steps = 1000; beta = 0.2
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)
```

```
*****
Iterations: 100 (20% complete)
Average Energy: -2.64
Average Magnetization: 0.2677777777777778
*****
Iterations: 200 (20% complete)
Average Energy: -2.94
Average Magnetization: 0.8455555555555557
*****
Iterations: 300 (30% complete)
Average Energy: -4.173333333333333
Average Magnetization: 0.6581851851851843
*****
Iterations: 400 (40% complete)
Average Energy: -4.74
Average Magnetization: 0.29566666666666675
*****
Iterations: 500 (50% complete)
Average Energy: -4.424
Average Magnetization: 0.11300000000000002
*****
Iterations: 600 (60% complete)
Average Energy: -3.8066666666666666
Average Magnetization: 0.8625925925925926
*****
Iterations: 700 (70% complete)
Average Energy: -3.8628754287543
Average Magnetization: 0.8753333333333339
*****
Iterations: 800 (80% complete)
Average Energy: -4.8
Average Magnetization: 0.8966666666666662
*****
Iterations: 900 (90% complete)
Average Energy: -4.111111111111111
Average Magnetization: 0.884975388641975
*****
Iterations: 1000 (100% complete)
Average Energy: -4.148
Average Magnetization: 0.8548888888888887
*****
```



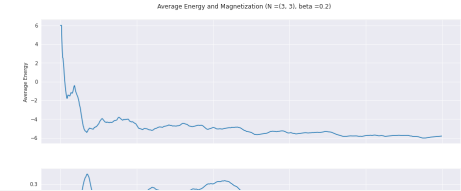
```
1 N = (3,3); steps = 1000; beta = 0.2
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)
```

```
*****
Iterations: 100 (20% complete)
Average Energy: -3.64
Average Magnetization: 0.1868888888888889
*****
Iterations: 200 (20% complete)
Average Energy: -5.1
Average Magnetization: 0.8811111111111116
*****
Iterations: 300 (30% complete)
Average Energy: -6.12
Average Magnetization: 0.1629629629629632
*****
Iterations: 400 (40% complete)
Average Energy: -6.29
Average Magnetization: 0.19855555555555554
*****
Iterations: 500 (50% complete)
Average Energy: -5.392
Average Magnetization: 0.1822222222222222
*****
Iterations: 600 (60% complete)
Average Energy: -5.133333333333333
Average Magnetization: 0.8625925925925926
*****
Iterations: 700 (70% complete)
Average Energy: -4.814285714285714
Average Magnetization: 0.8818475184751848
*****
Iterations: 800 (80% complete)
Average Energy: -4.89
Average Magnetization: 0.8788888888888888
*****
Iterations: 900 (90% complete)
Average Energy: -4.186666666666667
Average Magnetization: 0.8639801728395053
*****
Iterations: 1000 (100% complete)
Average Energy: -4.512
Average Magnetization: 0.8634646464646464
*****
```

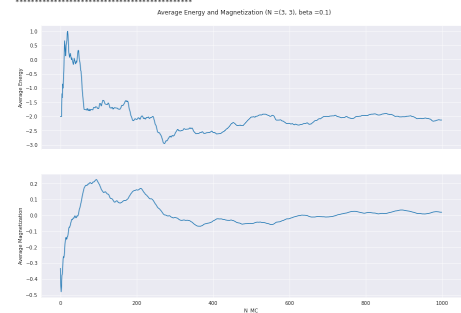


```
1 N = (3,3); steps = 1000; beta = 0.2
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)
```

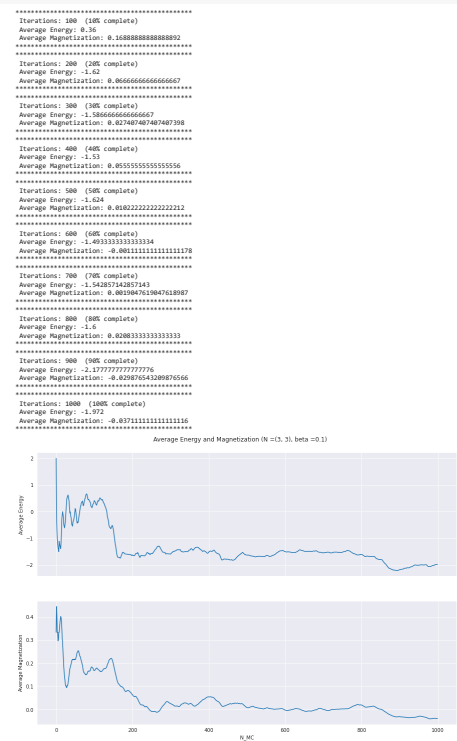
```
Iterations: 100 (10% complete)
Average Energy: -4.56
Average Magnetization: 0.21555555555555553
.....
Iterations: 200 (20% complete)
Average Energy: -4.58
Average Magnetization: 0.20333333333333342
.....
Iterations: 300 (30% complete)
Average Energy: -4.713333333333333
Average Magnetization: 0.2362962962962964
.....
Iterations: 400 (40% complete)
Average Energy: -4.51
Average Magnetization: 0.30055555555555554
.....
Iterations: 500 (50% complete)
Average Energy: -5.408
Average Magnetization: 0.20222222222222223
.....
Iterations: 600 (60% complete)
Average Energy: -5.48
Average Magnetization: 0.1592592592592583
.....
Iterations: 700 (70% complete)
Average Energy: -5.33742857142857
Average Magnetization: 0.16464646464646463
.....
Iterations: 800 (80% complete)
Average Energy: -5.7
Average Magnetization: 0.17240999999999998
.....
Iterations: 900 (90% complete)
Average Energy: -5.737777777777778
Average Magnetization: 0.119738864197382
.....
Iterations: 1000 (100% complete)
Average Energy: -5.832
Average Magnetization: 0.14911111111111109
.....
Average Energy and Magnetization (N =13, 3), beta =0.2)
```



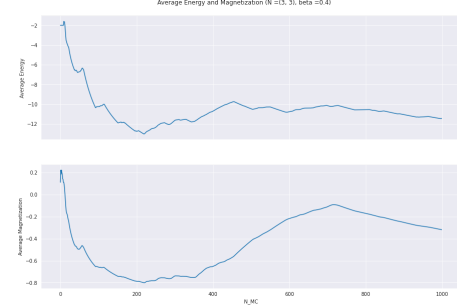
```
Iterations: 100 (10% complete)
Average Energy: -1.72
Average Magnetization: 0.2066666666666672
.....
Iterations: 200 (20% complete)
Average Energy: -2.1
Average Magnetization: 0.16111111111111112
.....
Iterations: 300 (30% complete)
Average Energy: -2.64
Average Magnetization: 0.01807407407407408
.....
Iterations: 400 (40% complete)
Average Energy: -2.55
Average Magnetization: 0.03666666666666668
.....
Iterations: 500 (50% complete)
Average Energy: -2.804
Average Magnetization: 0.05111111111111115
.....
Iterations: 600 (60% complete)
Average Energy: -2.24
Average Magnetization: 0.024814814814814815
.....
Iterations: 700 (70% complete)
Average Energy: -2.8
Average Magnetization: 0.009841269841269856
.....
Iterations: 800 (80% complete)
Average Energy: -3.505
Average Magnetization: 0.014722222222222226
.....
Iterations: 900 (90% complete)
Average Energy: -2.808888888888889
Average Magnetization: 0.02395857263951
.....
Iterations: 1000 (100% complete)
Average Energy: -2.124
Average Magnetization: 0.02
.....
Average Energy and Magnetization (N =13, 3), beta =0.1)
```



```
Iterations: 1800 (180% complete)
Average Energy: -2.68
Average Magnetization: -0.16666666666666666
Iterations: 200 (20% complete)
Average Energy: -2.84
Average Magnetization: -0.11666666666666667
Iterations: 400 (40% complete)
Average Energy: -2.53
Average Magnetization: -0.8555555555555556
Iterations: 600 (60% complete)
Average Energy: -1.654
Average Magnetization: -0.4062222222222222
Iterations: 800 (80% complete)
Average Energy: -1.4933333333333334
Average Magnetization: -0.8611111111111111
Iterations: 1000 (100% complete)
Average Energy: -1.543574287343
Average Magnetization: -0.8019847613047618
Iterations: 1200 (120% complete)
Average Energy: -1.6
Average Magnetization: -0.6266666666666667
Iterations: 1400 (140% complete)
Average Energy: -2.1777777777777776
Average Magnetization: -0.8208754320875666
Iterations: 1600 (160% complete)
Average Energy: -3.372
Average Magnetization: -0.8371111111111111
Iterations: 1800 (180% complete)
Average Energy: -3.372
Average Magnetization: -0.8371111111111111
Average Energy and Magnetization (N = (3, 3), beta = 0.1)
```



```
Iterations: 1800 (180% complete)
Average Energy: -10.14
Average Magnetization: -0.6577777777777779
Iterations: 200 (20% complete)
Average Energy: -12.72
Average Magnetization: -0.7866666666666666
Iterations: 400 (40% complete)
Average Energy: -10.72
Average Magnetization: -0.6518888888888887
Iterations: 600 (60% complete)
Average Energy: -10.448
Average Magnetization: -0.4222222222222222
Iterations: 800 (80% complete)
Average Energy: -10.766666666666667
Average Magnetization: -0.3188888888888888
Iterations: 1000 (100% complete)
Average Energy: -10.11435742872
Average Magnetization: -0.11206349206349206
Iterations: 1200 (120% complete)
Average Energy: -10.35
Average Magnetization: -0.1762777777777779
Iterations: 1400 (140% complete)
Average Energy: -11.864444444444444
Average Magnetization: -0.2523456790234565
Iterations: 1600 (160% complete)
Average Energy: -11.452
Average Magnetization: -0.3184444444444444
Iterations: 1800 (180% complete)
Average Energy: -11.452
Average Magnetization: -0.3184444444444444
Average Energy and Magnetization (N = (3, 3), beta = 0.4)
```



```
Iterations: 1800 (180% complete)
Average Energy: -11.452
Average Magnetization: -0.3184444444444444
Iterations: 200 (20% complete)
Average Energy: -12.72
Average Magnetization: -0.7866666666666666
Iterations: 400 (40% complete)
Average Energy: -10.72
Average Magnetization: -0.6518888888888887
Iterations: 600 (60% complete)
Average Energy: -10.448
Average Magnetization: -0.4222222222222222
Iterations: 800 (80% complete)
Average Energy: -10.766666666666667
Average Magnetization: -0.3188888888888888
Iterations: 1000 (100% complete)
Average Energy: -10.11435742872
Average Magnetization: -0.11206349206349206
Iterations: 1200 (120% complete)
Average Energy: -10.35
Average Magnetization: -0.1762777777777779
Iterations: 1400 (140% complete)
Average Energy: -11.864444444444444
Average Magnetization: -0.2523456790234565
Iterations: 1600 (160% complete)
Average Energy: -11.452
Average Magnetization: -0.3184444444444444
Iterations: 1800 (180% complete)
Average Energy: -11.452
Average Magnetization: -0.3184444444444444
Average Energy and Magnetization (N = (3, 3), beta = 0.4)
```

```

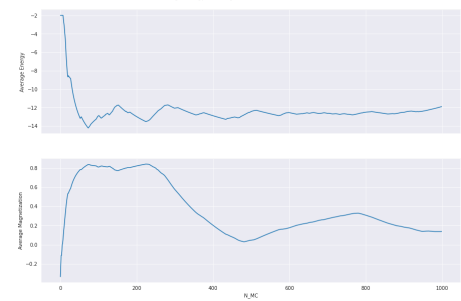
Iterations: 500 (100% complete)
Average Energy: -15.64
Average Magnetization: 0.90
=====
Iterations: 200 (20% complete)
Average Energy: -15.22
Average Magnetization: 0.8999999999999999
=====
Iterations: 300 (30% complete)
Average Energy: -13.986666666666666
Average Magnetization: 0.4581851851851866
=====
Iterations: 400 (40% complete)
Average Energy: -10.32
Average Magnetization: 0.4156666666666667
=====
Iterations: 500 (50% complete)
Average Energy: -10.68
Average Magnetization: 0.21555555555555554
=====
Iterations: 600 (60% complete)
Average Energy: -11.18
Average Magnetization: 0.04444444444444445
=====
Iterations: 700 (70% complete)
Average Energy: -12.154285714285715
Average Magnetization: -0.08761047110467
=====
Iterations: 800 (80% complete)
Average Energy: -11.685
Average Magnetization: -0.07222222222222222
=====
Iterations: 900 (90% complete)
Average Energy: -11.633333333333334
Average Magnetization: -0.08617203958617283
=====
1 N = (3,3); steps = 1000; beta = 0.4
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

```

```

=====
Iterations: 100 (10% complete)
Average Energy: -12.12
Average Magnetization: 0.8888888888888887
=====
Iterations: 200 (20% complete)
Average Energy: -12.36
Average Magnetization: 0.8288888888888887
=====
Iterations: 300 (30% complete)
Average Energy: -12.666666666666666
Average Magnetization: 0.57818518518517
=====
Iterations: 400 (40% complete)
Average Energy: -12.17
Average Magnetization: 0.20499999999999995
=====
Iterations: 500 (50% complete)
Average Energy: -12.448
Average Magnetization: 0.04666666666666667
=====
Iterations: 600 (60% complete)
Average Energy: -12.54
Average Magnetization: 0.1751851851851848
=====
Iterations: 700 (70% complete)
Average Energy: -12.174285714285714
Average Magnetization: 0.2098420984209
=====
Iterations: 800 (80% complete)
Average Energy: -12.585
Average Magnetization: 0.38811111111111096
=====
Iterations: 900 (90% complete)
Average Energy: -12.155555555555556
Average Magnetization: 0.1841971086410688
=====
Iterations: 1000 (100% complete)
Average Energy: -11.3
Average Magnetization: 0.13777777777777778
=====
Average Energy and Magnetization (N =(3, 3), beta =0.4)

```



```

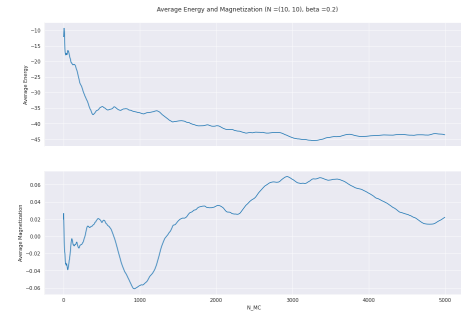
1 N = (10,10); steps = 5000; beta = 0.2
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

```

```

=====
Iterations: 500 (10% complete)
Average Energy: -34.24
Average Magnetization: 0.6547999999999999
=====
Iterations: 2000 (20% complete)
Average Energy: -36.104
Average Magnetization: 0.07400000000000002
=====
Iterations: 1500 (30% complete)
Average Energy: -39.176
Average Magnetization: 0.61743999999999997
=====
Iterations: 2000 (40% complete)
Average Energy: -40.742
Average Magnetization: 0.07500000000000002
=====
Iterations: 2500 (50% complete)
Average Energy: -42.84
Average Magnetization: 0.048160000000000034
=====
Iterations: 3000 (60% complete)
Average Energy: -46.406
Average Magnetization: 0.05420000000000002
=====
Iterations: 3500 (70% complete)
Average Energy: -44.38154285714286
Average Magnetization: 0.063973218714381
=====
Iterations: 4000 (80% complete)
Average Energy: -44.811
Average Magnetization: 0.05275000000000007
=====
Iterations: 4500 (90% complete)
Average Energy: -43.42133333333333
Average Magnetization: 0.02566222222222413
=====
Iterations: 5000 (100% complete)
Average Energy: -43.6848
Average Magnetization: 0.0212000000000148
=====
Average Energy and Magnetization (N =(10, 10), beta =0.2)

```

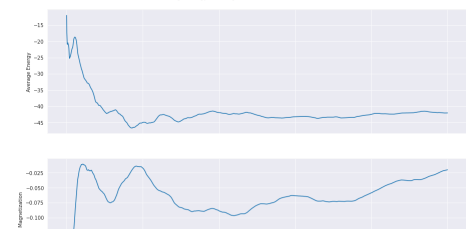


```

1 N = (10,10); steps = 5000; beta = 0.2
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

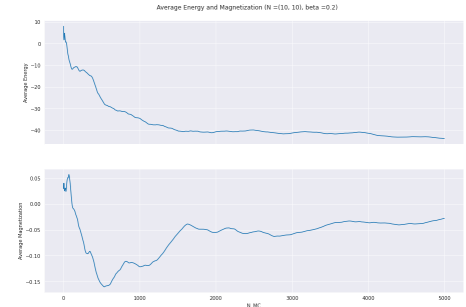
```

```
Iterations: 500 (10% complete)
Average Energy: -41.816
Average Magnetization: -0.8612000000000005
.....
Iterations: 1000 (20% complete)
Average Energy: -44.952
Average Magnetization: -0.8171000000000008
.....
Iterations: 1500 (30% complete)
Average Energy: -48.461333333333336
Average Magnetization: -0.86238000000000012
.....
Iterations: 2000 (40% complete)
Average Energy: -41.924
Average Magnetization: -0.80230000000000073
.....
Iterations: 2500 (50% complete)
Average Energy: -42.688
Average Magnetization: -0.87891999999999984
.....
Iterations: 3000 (60% complete)
Average Energy: -43.284
Average Magnetization: -0.86133999999999998
.....
Iterations: 3500 (70% complete)
Average Energy: -41.328
Average Magnetization: -0.874280742867132
.....
Iterations: 4000 (80% complete)
Average Energy: -42.438
Average Magnetization: -0.85459999999999999
.....
Iterations: 4500 (90% complete)
Average Energy: -42.803555555555556
Average Magnetization: -0.83821333333333349
.....
Iterations: 5000 (100% complete)
Average Energy: -42.8168
Average Magnetization: -0.83803600000000027
.....
Average Energy and Magnetization (N =10, 101, beta =0.2)
```



```
1 N = (10,10); steps = 5000; beta = 0.2
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)
```

```
Iterations: 500 (10% complete)
Average Energy: -25.688
Average Magnetization: -0.15839999999999996
.....
Iterations: 1000 (20% complete)
Average Energy: -34.434
Average Magnetization: -0.12128000000000017
.....
Iterations: 1500 (30% complete)
Average Energy: -40.320666666666666
Average Magnetization: -0.857439999999999825
.....
Iterations: 2000 (40% complete)
Average Energy: -48.652
Average Magnetization: -0.85488000000000001
.....
Iterations: 2500 (50% complete)
Average Energy: -39.8312
Average Magnetization: -0.85388799999999996
.....
Iterations: 3000 (60% complete)
Average Energy: -41.233333333333334
Average Magnetization: -0.85846666666666656
.....
Iterations: 3500 (70% complete)
Average Energy: -41.4697142871428
Average Magnetization: -0.83827142871428
.....
Iterations: 4000 (80% complete)
Average Energy: -41.393
Average Magnetization: -0.836179999999999914
.....
Iterations: 4500 (90% complete)
Average Energy: -43.887111111111111
Average Magnetization: -0.83869333333333368
.....
Iterations: 5000 (100% complete)
Average Energy: -41.784
Average Magnetization: -0.827675999999999885
.....
Average Energy and Magnetization (N =10, 101, beta =0.2)
```



```
1 N = (10,10); steps = 5000; beta = 0.1
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)
```

```
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)
```

```

Average Energy: -25.792
Average Magnetization: -0.08867999999999986
-----

```

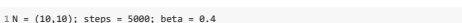
```
3 ds = Metropolis_Sampling(N,steps,beta)
```

```

*****
Iterations: 500 (10% complete)
Average Energy: -14.912

```

Average Magnetization: -0.05075999999999997  
\*\*\*\*\*



```

y03 = test_opoints_sampling(N, steps, seed

```

```

Iterations: 500 (100% complete)
Average Energy: -82.568
Average Magnetization: 0.10599999999999999
Iterations: 1000 (200% complete)
Average Energy: -82.476
Average Magnetization: 0.12640000000000002
Iterations: 1500 (300% complete)
Average Energy: -86.43866666666667
Average Magnetization: 0.40366666666666666
Iterations: 2000 (400% complete)
Average Energy: -106.922
Average Magnetization: 0.45430000000000002
Iterations: 2500 (500% complete)
Average Energy: -182.2472
Average Magnetization: 0.49134999999999995
Iterations: 3000 (600% complete)
Average Energy: -184.43866666666667
Average Magnetization: 0.5088133333333333
Iterations: 3500 (700% complete)
Average Energy: -182.216
Average Magnetization: 0.4681342857142857
Iterations: 4000 (800% complete)
Average Energy: -183.2
Average Magnetization: 0.45024999999999995
Iterations: 4500 (900% complete)
Average Energy: -182.33688888888888
Average Magnetization: 0.4414222222222223
Iterations: 5000 (1000% complete)
Average Energy: -185.1872
Average Magnetization: 0.45489999999999995
Average Energy and Magnetization (N =10, 10, beta =0.4)

```

```

1 N = (10,10); steps = 5000; beta = 0.4

```

```

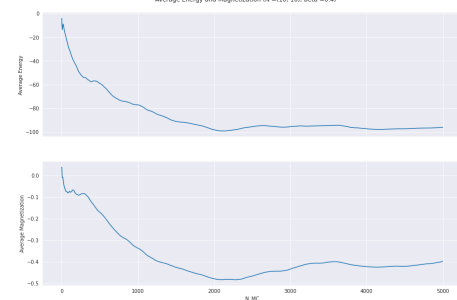
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

```

```

Iterations: 500 (100% complete)
Average Energy: -58.992
Average Magnetization: -0.17819999999999994
Iterations: 1000 (200% complete)
Average Energy: -77.128
Average Magnetization: -0.33720000000000007
Iterations: 1500 (300% complete)
Average Energy: -91.18666666666667
Average Magnetization: -0.43293333333333336
Iterations: 2000 (400% complete)
Average Energy: -98.128
Average Magnetization: -0.47849999999999993
Iterations: 2500 (500% complete)
Average Energy: -95.5664
Average Magnetization: -0.4668200000000001
Iterations: 3000 (600% complete)
Average Energy: -95.136
Average Magnetization: -0.4342666666666667
Iterations: 3500 (700% complete)
Average Energy: -98.458
Average Magnetization: -0.4821885714285714
Iterations: 4000 (800% complete)
Average Energy: -97.31
Average Magnetization: -0.42299000000000004
Iterations: 4500 (900% complete)
Average Energy: -97.36977777777778
Average Magnetization: -0.42146666666666676
Iterations: 5000 (1000% complete)
Average Energy: -95.3254
Average Magnetization: -0.39921600000000003
Average Energy and Magnetization (N =10, 10, beta =0.4)

```



```

1 N = (10,10); steps = 5000; beta = 0.4

```

```

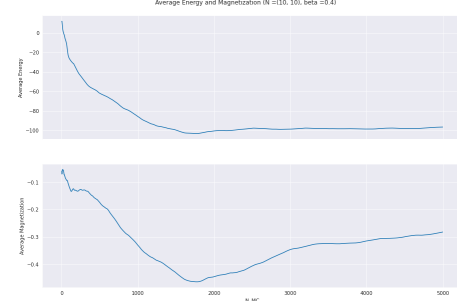
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

```

```

Iterations: 500 (100% complete)
Average Energy: -83.8
Average Magnetization: -0.17487999999999997
Iterations: 1000 (200% complete)
Average Energy: -85.42
Average Magnetization: -0.18680000000000006
Iterations: 1500 (300% complete)
Average Energy: -89.53333333333333
Average Magnetization: -0.43680000000000003
Iterations: 2000 (400% complete)
Average Energy: -106.398
Average Magnetization: -0.44515999999999998
Iterations: 2500 (500% complete)
Average Energy: -97.4272
Average Magnetization: -0.48617999999999997
Iterations: 3000 (600% complete)
Average Energy: -98.124
Average Magnetization: -0.34619999999999996
Iterations: 3500 (700% complete)
Average Energy: -98.40374285714286
Average Magnetization: -0.32375714285714285
Iterations: 4000 (800% complete)
Average Energy: -98.454
Average Magnetization: -0.34472000000000006
Iterations: 4500 (900% complete)
Average Energy: -97.31466666666666
Average Magnetization: -0.3954222222222223
Iterations: 5000 (1000% complete)
Average Energy: -96.3992
Average Magnetization: -0.28215999999999994
Average Energy and Magnetization (N =10, 10, beta =0.4)

```



```

1 N = (64,64); steps = 50000; beta = 0.2

```

```

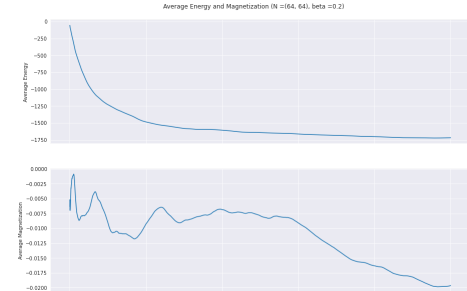
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

```

```

=====
Iterations: 5000 (10% complete)
Average Energy: -1230.284
Average Magnetization: -0.0086226021875
=====
Iterations: 10000 (20% complete)
Average Energy: -1486.8364
Average Magnetization: -0.00974853515625
=====
Iterations: 15000 (30% complete)
Average Energy: -1581.7429333333334
Average Magnetization: -0.00878143221666667
=====
Iterations: 20000 (40% complete)
Average Energy: -1698.2398
Average Magnetization: -0.008077662189375
=====
Iterations: 25000 (50% complete)
Average Energy: -1644.86244
Average Magnetization: -0.0098736328125
=====
Iterations: 30000 (60% complete)
Average Energy: -1644.3397333333333
Average Magnetization: -0.009108032512083333
=====
Iterations: 35000 (70% complete)
Average Energy: -1634.683428174286
Average Magnetization: -0.0155633958892856
=====
Iterations: 40000 (80% complete)
Average Energy: -1783.2142
Average Magnetization: -0.016169921875
=====
Iterations: 45000 (90% complete)
Average Energy: -1728.4512444444444
Average Magnetization: -0.0121274954597222
=====
Iterations: 50000 (100% complete)
Average Energy: -1728.39704
Average Magnetization: -0.01065640814375
=====
Average Energy and Magnetization (N = 64, beta = 0.2)

```



```

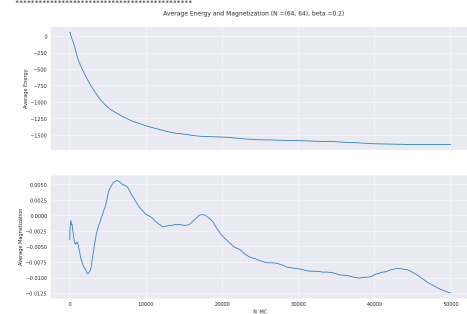
1 N = (64,64); steps = 50000; beta = 0.2
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

```

```

=====
Iterations: 5000 (10% complete)
Average Energy: -1877.428
Average Magnetization: 0.00348039621
=====
Iterations: 10000 (20% complete)
Average Energy: -1317.832
Average Magnetization: 0.00025712421875
=====
Iterations: 15000 (30% complete)
Average Energy: -1486.419
Average Magnetization: -0.001556755288333334
=====
Iterations: 20000 (40% complete)
Average Energy: -1525.8888
Average Magnetization: -0.0012823816448625
=====
Iterations: 25000 (50% complete)
Average Energy: -1572.8366
Average Magnetization: -0.0026787189375
=====
Iterations: 30000 (60% complete)
Average Energy: -1581.8586666666668
Average Magnetization: -0.008512796223958333
=====
Iterations: 35000 (70% complete)
Average Energy: -1685.8688
Average Magnetization: -0.0057269162679571
=====
Iterations: 40000 (80% complete)
Average Energy: -1632.2179
Average Magnetization: -0.00528348912189375
=====
Iterations: 45000 (90% complete)
Average Energy: -1643.3234444444444
Average Magnetization: -0.009133878472222222
=====
Iterations: 50000 (100% complete)
Average Energy: -1643.55936
Average Magnetization: -0.01242711953125
=====
Average Energy and Magnetization (N = 64, beta = 0.2)

```



```

1 N = (64,64); steps = 50000; beta = 0.2
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

```

C

```

Iterations: 5000 (50% complete)
Average Energy: -1231.2456
Average Magnetization: -0.8184709828125
=====
Iterations: 10000 (20% complete)
Average Energy: -1459.898
Average Magnetization: -0.8219287189375
=====
Iterations: 15000 (30% complete)
Average Energy: -1585.2773333333334
Average Magnetization: -0.82056370442708333
=====
Iterations: 20000 (40% complete)
Average Energy: -1650.3782
Average Magnetization: -0.820302080078125
=====
Iterations: 25000 (50% complete)
Average Energy: -1650.8852
Average Magnetization: -0.8204729828125
=====

```

```

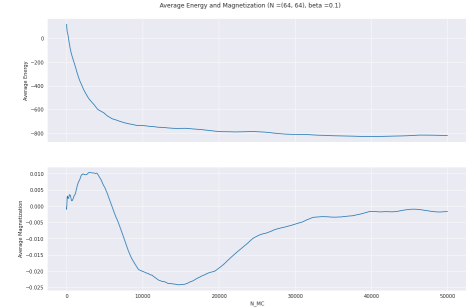
1 N = (64,64); steps = 50000; beta = 0.1
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

```

```

=====
Iterations: 5000 (10% complete)
Average Energy: -634.32
Average Magnetization: 0.00570801984375
=====
Iterations: 10000 (20% complete)
Average Energy: -735.8724
Average Magnetization: -0.80265265546875
=====
Iterations: 15000 (30% complete)
Average Energy: -755.2373333333334
Average Magnetization: -0.824132773947916665
=====
Iterations: 20000 (40% complete)
Average Energy: -790.8644
Average Magnetization: -0.80841259765625
=====
Iterations: 25000 (50% complete)
Average Energy: -787.8648
Average Magnetization: -0.80928208406875
=====
Iterations: 30000 (60% complete)
Average Energy: -805.4536666666667
Average Magnetization: -0.805534716796875
=====
Iterations: 35000 (70% complete)
Average Energy: -821.47428071428
Average Magnetization: -0.8038454708234286
=====
Iterations: 40000 (80% complete)
Average Energy: -828.5127
Average Magnetization: -0.8059893798828125
=====
Iterations: 45000 (90% complete)
Average Energy: -828.8718222222222
Average Magnetization: -0.8080811762127777
=====
Iterations: 50000 (100% complete)
Average Energy: -828.25312
Average Magnetization: -0.8062375398625
=====

```



```

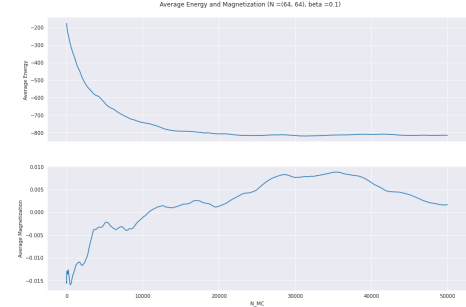
1 N = (64,64); steps = 50000; beta = 0.1
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

```

```

=====
Iterations: 5000 (10% complete)
Average Energy: -631.6152
Average Magnetization: 0.00461238125
=====
Iterations: 10000 (20% complete)
Average Energy: -741.2584
Average Magnetization: -0.8001406134375
=====
Iterations: 15000 (30% complete)
Average Energy: -791.2883333333334
Average Magnetization: 0.0018994148625
=====
Iterations: 20000 (40% complete)
Average Energy: -807.4784
Average Magnetization: 0.00136367421875
=====
Iterations: 25000 (50% complete)
Average Energy: -817.3264
Average Magnetization: 0.00499408398625
=====
Iterations: 30000 (60% complete)
Average Energy: -817.5556
Average Magnetization: 0.0076580280241667
=====
Iterations: 35000 (70% complete)
Average Energy: -814.223774285715
Average Magnetization: 0.0067718452671428
=====
Iterations: 40000 (80% complete)
Average Energy: -818.265
Average Magnetization: 0.0063572265625
=====
Iterations: 45000 (90% complete)
Average Energy: -815.5233333333334
Average Magnetization: 0.00385451388888888
=====
Iterations: 50000 (100% complete)
Average Energy: -815.41392
Average Magnetization: 0.004551788984375
=====

```



```

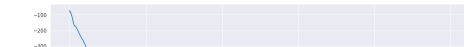
1 N = (64,64); steps = 50000; beta = 0.1
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

```

```

Iterations: 5000 (100% complete)
Average Energy: -595.7288
Average Magnetization: -0.407434815625
=====
Iterations: 10000 (200% complete)
Average Energy: -656.4128
Average Magnetization: -0.400645894375
=====
Iterations: 15000 (300% complete)
Average Energy: -685.133666666667
Average Magnetization: -0.40021767578125
=====
Iterations: 20000 (400% complete)
Average Energy: -721.771
Average Magnetization: -0.404957439653125
=====
Iterations: 25000 (500% complete)
Average Energy: -742.4748
Average Magnetization: -0.4016669765625
=====
Iterations: 30000 (600% complete)
Average Energy: -750.1828
Average Magnetization: -0.401282467916666666
=====
Iterations: 35000 (700% complete)
Average Energy: -751.13285742858
Average Magnetization: -0.4005414815625
=====
Iterations: 40000 (800% complete)
Average Energy: -767.6268
Average Magnetization: -0.400628936948625
=====
Iterations: 45000 (900% complete)
Average Energy: -785.134933333334
Average Magnetization: -0.40038648497985111
=====
Iterations: 50000 (1000% complete)
Average Energy: -795.46888
Average Magnetization: -0.4005964641125
=====
Average Energy and Magnetization (N =64, 64), beta =0.1)

```



```

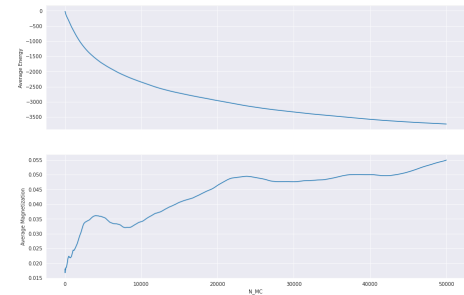
1 N = (64,64); steps = 50000; beta = 0.4
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

```

```

Iterations: 5000 (100% complete)
Average Energy: -1752.7984
Average Magnetization: -0.8355794815625
=====
Iterations: 10000 (200% complete)
Average Energy: -1345.2136
Average Magnetization: -0.83495886571875
=====
Iterations: 15000 (300% complete)
Average Energy: -2713.13333333333
Average Magnetization: -0.400639897916667
=====
Iterations: 20000 (400% complete)
Average Energy: -2950.514
Average Magnetization: -0.404796978125
=====
Iterations: 25000 (500% complete)
Average Energy: -3375.3344
Average Magnetization: -0.40482126796875
=====
Iterations: 30000 (600% complete)
Average Energy: -3334.82933333334
Average Magnetization: -0.407384141833334
=====
Iterations: 35000 (700% complete)
Average Energy: -3459.167085742855
Average Magnetization: -0.4047613098667425
=====
Iterations: 40000 (800% complete)
Average Energy: -3375.2286
Average Magnetization: -0.40497272728515625
=====
Iterations: 45000 (900% complete)
Average Energy: -3653.49848888889
Average Magnetization: -0.4058971371522778
=====
Iterations: 50000 (1000% complete)
Average Energy: -3786.8626
Average Magnetization: -0.4049241815625
=====
Average Energy and Magnetization (N =64, 64), beta =0.4)

```



```

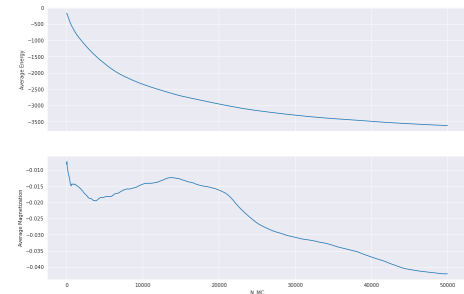
1 N = (64,64); steps = 50000; beta = 0.4
2
3 ds = Metropolis_Sampling(N,steps,beta); Make_Plots(ds)

```

```

Iterations: 5000 (100% complete)
Average Energy: -1717.8472
Average Magnetization: -0.8343849451125
=====
Iterations: 10000 (200% complete)
Average Energy: -2348.7802
Average Magnetization: -0.83438683515625
=====
Iterations: 15000 (300% complete)
Average Energy: -2787.817866666666
Average Magnetization: -0.812894899591333
=====
Iterations: 20000 (400% complete)
Average Energy: -2955.1478
Average Magnetization: -0.8161894138859375
=====
Iterations: 25000 (500% complete)
Average Energy: -3318.6212
Average Magnetization: -0.8263666189375
=====
Iterations: 30000 (600% complete)
Average Energy: -3298.67613333334
Average Magnetization: -0.8388976171875
=====
Iterations: 35000 (700% complete)
Average Energy: -3448.16574287144
Average Magnetization: -0.8313958597853571
=====
Iterations: 40000 (800% complete)
Average Energy: -3489.2688
Average Magnetization: -0.8368975788078125
=====
Iterations: 45000 (900% complete)
Average Energy: -3563.18622222222
Average Magnetization: -0.84072768815127776
=====
Iterations: 50000 (1000% complete)
Average Energy: -3615.252
Average Magnetization: -0.84289724689375
=====
Average Energy and Magnetization (N =64, 64), beta =0.4)

```



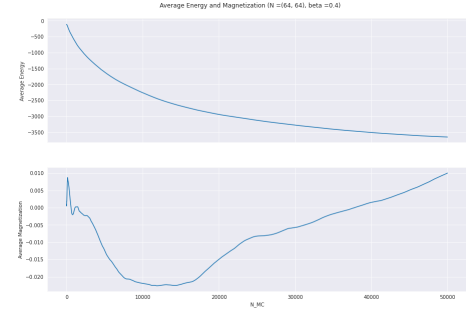
```

1 N = (64,64); steps = 50000; beta = 0.4

```

ds = Metropolis\_Sampling(N,steps,beta); Make\_Plots(ds)

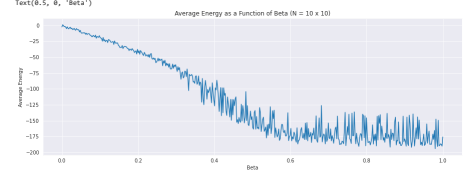
```
Iterations: 5000 (10% complete)
Average Energy: -842.545
Average Magnetization: -0.812414515078125
*****
Iterations: 10000 (20% complete)
Average Energy: -2275.8412
Average Magnetization: -0.82195218700625
*****
Iterations: 15000 (30% complete)
Average Energy: -2675.4984
Average Magnetization: -0.8224524759583334
*****
Iterations: 20000 (40% complete)
Average Energy: -2940.5023
Average Magnetization: -0.8149788338078125
*****
Iterations: 25000 (50% complete)
Average Energy: -3124.3554
Average Magnetization: -0.8082380148625
*****
Iterations: 30000 (60% complete)
Average Energy: -3277.377066666665
Average Magnetization: -0.8057795154666667
*****
Iterations: 35000 (70% complete)
Average Energy: -3484.488285714285
Average Magnetization: -0.801882646285357142
*****
Iterations: 40000 (80% complete)
Average Energy: -3587.4736
Average Magnetization: -0.801530955876125
*****
Iterations: 45000 (90% complete)
Average Energy: -3588.625777777778
Average Magnetization: -0.8054068484777778
*****
Iterations: 50000 (100% complete)
Average Energy: -3645.32936
Average Magnetization: -0.81082280854875
*****
Average Energy and Magnetization (N = 54, 64, beta = 0.4)
```



• Runs with a  $10 \times 10$  matrix at variable  $\beta$ :

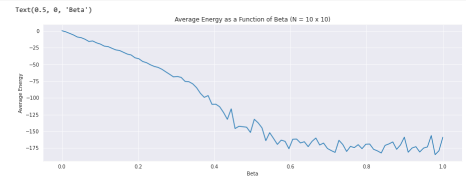
```
1 N = (10,10); steps = 5000;
2
3 entries = 500; E_fb = []
4
5 #Suppressing the algorithms printouts.
6 sys.stdout = open(os.devnull, 'w')
7 for value in range(0,entries,1):
8     #Runs the algorithm for each value of beta in this range
9     ds = Metropolis_Sampling(N,steps,value/entries)
10    #Records an "average" average energy to help smooth out the data.
11    E_fb.append(np.average(ds["bar_E"][-100:]))
12 sys.stdout = sys._stdout__
13
```

```
1 plt.figure(figsize=(15,5))
2 plt.plot(np.linspace(0,1,entries),E_fb)
3 plt.title("Average Energy as a Function of Beta (N = 10 x 10)",)
4 plt.ylabel("Average Energy"); plt.xlabel("Beta")
```



```
1 entries = 100; E_fb = []
2
3 #This is very similar to the code above
4 #We are using less values for beta but taking a sample at each value.
5
6 sys.stdout = open(os.devnull, 'w')
7 for value in range(0,entries,1):
8     running_average = 0
9     for _ in range(5):
10        ds = Metropolis_Sampling(N,steps,value/entries)
11        running_average += np.average(ds["bar_E"][-100:])
12    E_fb.append(running_average/5)
13 sys.stdout = sys._stdout__
```

```
1 plt.figure(figsize=(15,5))
2 plt.plot(np.linspace(0,1,entries),E_fb)
3 plt.title("Average Energy as a Function of Beta (N = 10 x 10)",)
4 plt.ylabel("Average Energy"); plt.xlabel("Beta")
```



```
1
2
3 entries = 100;
4 E_fb = []; M_fb = []
5
6 #This is very similar to the code above
7 #We are using less values for beta but taking a sample at each value.
8
9 sys.stdout = open(os.devnull, 'w')
10 for value in range(0,entries,1):
11     running_average = 0
12     M_temp = ()
13     for _ in range(5):
14        ds = Metropolis_Sampling(N,steps,value/entries)
15        running_average += np.average(ds["bar_E"][-100:])
16        M_temp.append(ds["bar_M"][-1])
17    E_fb.append(running_average/5)
18    M_fb.append(M_temp)
19 sys.stdout = sys._stdout__
```

```
1 plt.figure(figsize=(15,5))
2 plt.plot(np.linspace(0,1,entries),E_fb)
3 plt.title("Average Energy as a Function of Beta (N = 10 x 10)",)
4 plt.ylabel("Average Energy"); plt.xlabel("Beta")
```

